Computational Geology 31

Visualizing averages – The 60% Relative Depth Rule for Stream Velocity

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Topics this issue-

Mathematics: average; weighted average; numerical integration.
Geology: Velocity profile in a stream.

Given the theme of this special issue – how we think about geological subjects – it may be appropriate for me to tell a story about the perception, and common misconception, of an average that appears in a routine hydrological procedure. The story involves the variation of flow velocity with depth in a stream.

The key contextual point of the story is that, although flow velocity varies from top to bottom through the depth of the stream (Figure 1), the average along a vertical line at any geographic position along the stream lies rather consistently at 60% of the stream depth (i.e., a relative depth of 0.6) (e.g., Leopold et al., 1964, Figure 6-3; Daugherty et al., 1985, Figure 11.6; Gordon et al., 1992, Table 5.1). This point is important in field hydrology because it simplifies the quick determination of discharge (Q) in the stream. The standard procedure for shallow streams performed in nearly every hydrology course is to determine the average stream velocity (v_{ave}) at each of several vertical slices lined up across the stream, multiply each of those average velocities by the area of the corresponding slice (A_i) to obtain the discharge through each of the slices ($Q_i = v_{ave}A_i$), and then sum the slice discharges ($Q = \Sigma Q_i$). The method is a version of numerical integration performed in the field. The "0.6 rule" comes in because it obviates the need to measure the velocity profile, $v(D)$ with D being depth, in each of the vertical slices.

The point to the story is that an average might not be what one visualizes it to be while looking at a graph. I mean, one has to take care and think about the graph, and at the same time, think about the meaning of the average.

THE QUESTION

The story starts with a conversation between two students. The students were recalling the 0.6 rule that they learned in prior classes. They needed to know how to measure the stream discharge in order to do a project. I listened in on their conversation.

The students recalled that the average velocity along a vertical is reputed to lie at a relative depth of 0.6. They disagreed, however, on the direction of the 0.6. That is, is the average velocity at a depth of 60% of the total depth as measured from the top of the stream (i.e., below the water level)? Or is it 60% of the total depth as measured up from the bottom of the stream (i.e., above the streambed). In terms of a diagram, is the average velocity as shown in Figure 2A (0.6 measured from the top) or Figure 2B (0.6 measured from the bottom)?

To their credit, the students tried to figure out the answer, rather than simply going to look it up. They sketched both diagrams, and then argued about which vector representing an average looked more reasonable. One student, with a strong chemistry background, thought about the numbers, and didn't see what the disagreement was about; she selected the equivalent of Figure 2A. Another student, with a strong geology background, took a more visually oriented approach. She asked: "With all those longer vectors high up in the graph, how could it not be that the average is higher than mid-depth?" – or words to that effect. Then the students took the problem to other students. The venue for the continuing discussion was a pub where they couldn't look up an answer. The discussion, I'm told, went on for quite a while, but no one's point of view changed.

Intrigued by the question and particularly about what people were thinking when they tried to answer it, I decided to poll a group of educators participating in a summer workshop on Quantitative Literacy. I showed them a slide containing the two options of Figure 2. Several of the participants and resource team members at the workshop were hydrologists, and they abstained from the poll. The others included faculty from mathematics, chemistry, social science, public health, and English. The results of the polls: Option A, 25%; Option B, 75%. Option A is the correct answer.

OPTION A

The theme of my presentation at the workshop was: When confronted with a question involving numbers, break out a spreadsheet and do a simple calculation. Thus Figure 3 shows a spreadsheet that calculates the average from Figure 1.

The spreadsheet works as follows. Columns B and C list the data. Columns D and E do the calculations. Column D lists the average of depths at two vertically adjoining cells; e.g., the value in Cell D4 is the average of the values in Cells B3 and B4. Column E lists the average of the velocity at the same two vertically adjoining cells; i.e., the value in Cell E4 is the average of the values in Cells C3 and C4. Cell E14 is the average of the values in column E. Because the total depth interval is 1 (Cell B13), this value in Cell E14 is the overall average velocity by the trapezoid rule (the area "under the curve" of velocity vs. depth divided by the length of the relative-depth axis). This average value (3.145 ft/sec) is the same as the velocity at relative depth 0.55 (3.15 ft/sec, Cells D9 and E9). Therefore, the average velocity in the curve is at 55% of the depth measured downward from the top of the stream.

Figure 4 shows a visual representation of the result. The average at 0.55 is indicated by the heavy vector at A. To say that this vector represents the average is to say that the area of BCA is equal to the area ADE, because area OBED must equal area OBCAD. In other words, the stack of left-to-right, horizontal vectors covering OBED must represent the same discharge as the stack of left-to-right, horizontal vectors covering OBCAD.

Figure 4 is a convincing argument that the average cannot occur at a relative depth of 0.4. If the line BE were shifted to the right to where point A lines up with relative

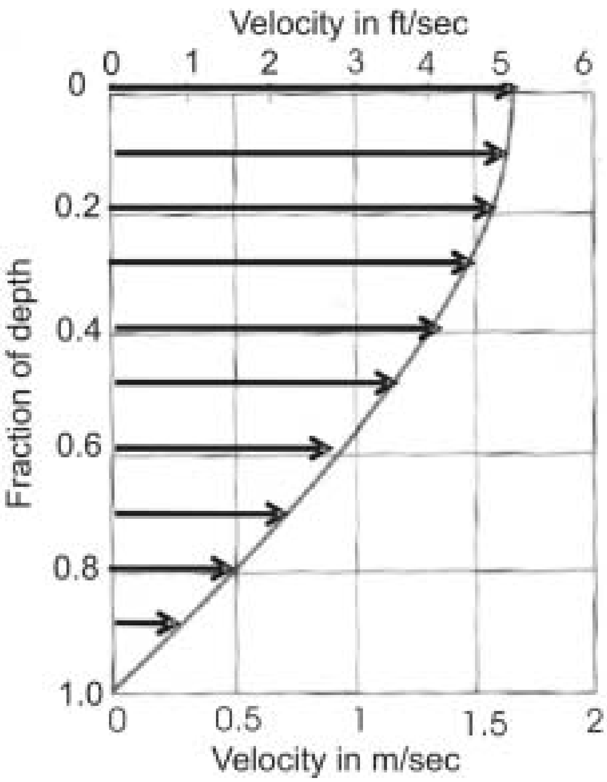

Figure 1. Typical graph of stream velocity vs. depth in a stream. Adapted from Linsley et al., 1982, Figure 4-8.

depth 0.4, the area BCA would be much smaller than the area ADE.

OPTION B

Now that the calculation is completed, and the visual argument for Option A is in hand, what are all those people thinking when they select Option B?

I have the benefit of the report back from the conversation at the pub. Based on that report, I made the following conjecture to the QL group that selected incorrect Option B over correct Option A. For what it is worth, the group agreed that, yes indeed, they were thinking what I conjectured.

Here is the conjecture.

The misconception is that the average of the long vectors and the short ones can be obtained by visually rotating the graph and then balancing the vectors on a see saw as in Figure 5.

How can we test that this approach gives the number that these people visualize? The answer, as usual, is to break out a spreadsheet and do a simple calculation. Figure 6 shows the calculation

The calculation is a weighted average: the average depth weighted by the length of the velocity vectors. Again Columns B and C list the data. Again, Columns D and E calculate the two-point, mid-level averages of depths and velocities, respectively. Column F calculates the mid-level products; e.g., Cell F4 is the product of Cells D4 and E4. Cell E15 is the sum of the mid-level products. Cell E16 is the sum of the mid-level velocities (the sum of the weights). Cell F17 is the weighted

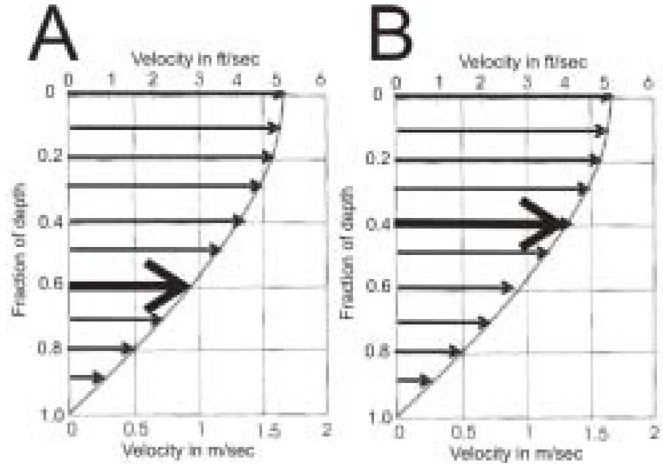

Figure 2. Which of these is the correct "O.6 rule"? A. The average velocity is located at a relative depth of 0.6 down from the top of the stream. B. The average velocity is located at a relative depth of 0.6 up from the bottom of the stream.

	B	C	D	E
2	depth	velocity	mid-depth	mid-vel
3	0	5.1		
4	0.1	5	0.05	5.05
5	0.2	4.8	0.15	4.9
6	0.3	4.4	0.25	4.6
7	0.4	4	0.35	4.2
8	0.5	3.5	0.45	3.75
9	0.6	2.8	0.55	3.15
10	0.7	2.2	0.65	2.5
11	0.8	1.5	0.75	1.85
12	0.9	0.7	0.85	1.1
13	1	0	0.95	0.35
14	overll avg			3.145

Figure 3. Spreadsheet calculating the average velocity from the profile shown in Figure 1.

average: the sum of the velocity-weighted mid-level depths divided by the sum of their weights. The answer, 0.36 for the relative depth, gives the location of the fulcrum for the sea saw of Figure 5.

DISCUSSION AND CONCLUSION

This anecdote suggests that, for many people, the mind's-eye mistakenly looks for the balance-point depth when seeking the depth of the average velocity. The balance-point depth is a weighted average: the average depth weighted by the velocities. That this is not the same as the depth at which the average velocity occurs can be shown relatively easily for a triangular distribution of velocity (Figure 7).

Figure 7 is a graph of velocity vs. depth rotated 90° counter-clockwise so that the independent variable (*D*,

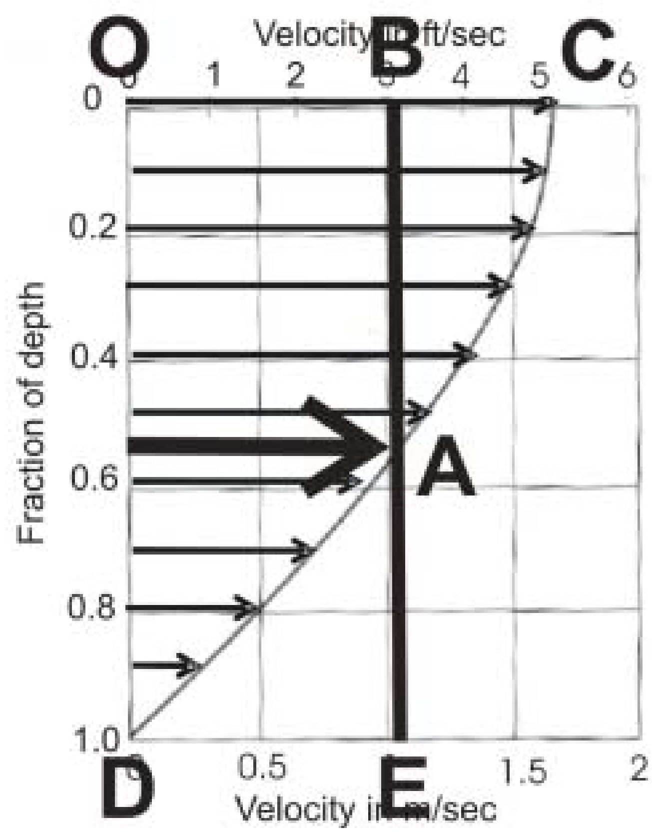

Figure 4. Sketch showing the a rectangular area OBED equal to the area OBCAD outlined by the velocity profile of Figure 1.

depth) increases downward on a vertical axis and the dependent variable (v, horizontal stream velocity) increases left to right on a horizontal axis. The function describing the velocity variation is

$$v = v_0\left(1 - \frac{D}{D_B}\right), \tag{1}$$

where v_0 is the maximum velocity at the top of the stream and D_B is the maximum depth of water at the bottom of the stream. Obviously, the average velocity is $v_0/2$ and occurs at a depth of $D_B/2$ (Figure 7). Where is the balance-point depth?

The balance-point depth is the velocity-weighted average depth or

$$\overline{D} = \frac{\int_0^{D_B} vDdD}{\int_0^{D_B} vdD}, \tag{2}$$

which follows from one's recollection of the weighted average as the weighted sum over the sum of the weights (from Physics 1):

$$\overline{x} = \frac{\sum_i w_i x_i}{\sum_i w_i}. \tag{3}$$

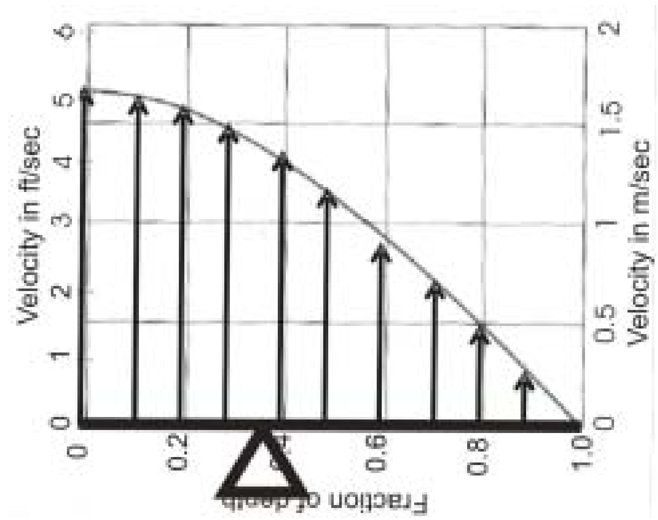

Figure 5. Sketch showing the balance-point depth (fulcrum) for the velocity profile in Figure 1.

	B	C	D	E	F
2	depth	velocity	mid-depth	mid-vel	D*E
3	0	5.1			
4	0.1	5	0.05	5.05	0.2525
5	0.2	4.8	0.15	4.9	0.735
6	0.3	4.4	0.25	4.6	1.15
7	0.4	4	0.35	4.2	1.47
8	0.5	3.5	0.45	3.75	1.6875
9	0.6	2.8	0.55	3.15	1.7325
10	0.7	2.2	0.65	2.5	1.625
11	0.8	1.5	0.75	1.85	1.3875
12	0.9	0.7	0.85	1.1	0.935
13	1	0	0.95	0.35	0.3325
14					
15	SumProduct				11.3075
16	Sum weights			31.45	
17	Avg depth weighted by velocites				0.360

Figure 6. Spreadsheet calculating the balance-point depth (fulcrum) shown in Figure 5.

Substituting Equation 1 into the numerator of Equation 2 leads to

$$\int_0^{D_B} vDdD = \frac{D_B^2}{6}. \tag{4}$$

Substituting Equation 1 into the denominator of Equation 2 leads to

$$\int_0^{D_B} vdD = \frac{D_B}{2}. \tag{5}$$

Combining Equations 2, 4 and 5 produces

$$\overline{D} = \frac{D_B}{3}. \tag{6}$$

As a check on the calculus, one can find the same answer by modifying the data in the spreadsheet in Figure 6 to do the calculation for a triangular velocity profile.

For the type of stream profile shown in Figure 1, the average velocity occurs at a relative depth of around 0.6.

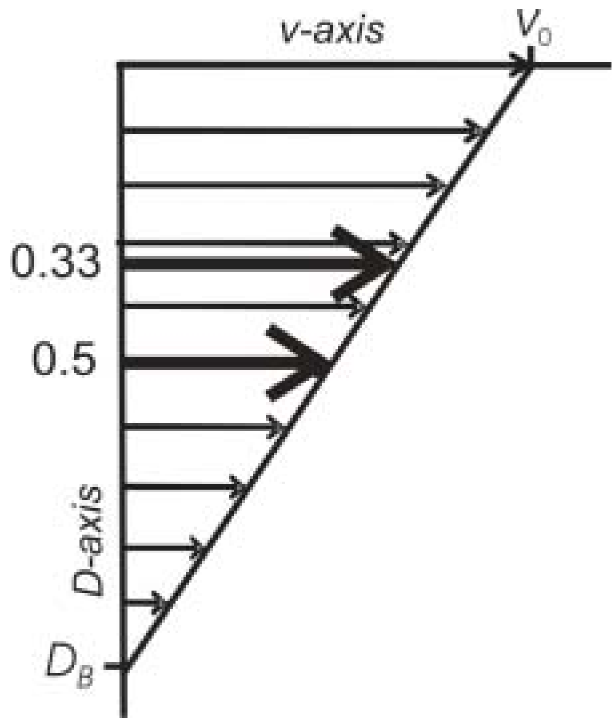

Figure 7. Graph of a hypothetical velocity profile in which velocity decreases linearly with depth.

In comparison, the average velocity occurs at a relative depth of 0.5 in the hypothetical case where velocity decreases linearly with depth. If one's mind eye is inclined to look for the average velocity at the "center of mass" of the velocity vectors, then one would perceive it higher in the profile – at a relative depth of around 0.4 in Figure 1 and at 0.33 in the hypothetical linear-variation case in figure 7.

If there were no variation of velocity with depth – i.e., if the velocity distribution were rectangular as in line BE of Figure 4 – then we would have the uninteresting situation where the relative depth of the average velocity would be the same as the velocity-weighted average depth. Both would be at mid-depth: 0.5.

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